Mathematics Methods
Unit 3 & 4

Integration

1. Indefinite integration rules

(a) Increase the power by one and divide by the new power

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c$$

Example:
Integrate $f'(x) = 2x$

$$\int 2x \; dx = \frac{2x^{1+1}}{1+1} + c$$
$$= x^2 + c$$

(b) Others

By substitution	By formula
$\int (ax+b)^n \, dx$	$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, n \neq 1$

Example:
$\int (5-3x)^2 \; dx$

By substitution	By formula
Let $u = 5-3x, \frac{du}{dx} = -3$ $\int (5-3x)^2 \; dx = \int u^2 \, (-\frac{1}{3} du)$ $= \left(-\frac{1}{3}\right)\left(\frac{u^3}{3}\right) + c$ $= -\frac{(5-3x)^3}{9} + c$	$\int (5-3x)^2 \; dx = \frac{(5-3x)^{2+1}}{(2+1)(-3)} + c$ $= -\frac{(5-3x)^3}{9} + c$

Trigonometric functions

$$\int \cos x \, dx = \sin x + c$$
$$\int \cos ax \, dx = \frac{1}{a} \sin x + c$$
$$\int \sin x \, dx = -\cos x + c$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos x\ + c$$

$$\int sec^2 x\ \ dx =\ tan\, x\ + c$$

Example 1:
Integrate $15\ cos\ 5x$.

$$\int 15\ cos\ 5x\ \ dx$$

Let $u = 5x,$

$\frac{du}{dx} = 5$

$dx = \frac{du}{5}$

$\int 15\ cos\ u\ \ \frac{du}{5}$

$=\ 3\ sin\ u + c$

$=\ 3\ sin\ 5x\ + c$

Example 2:
Integrate $sin\ 5x\ + 6x$

$$\int sin\ 5x\ + 6x\ dx$$

Let $u = 5x,$

$\frac{du}{dx} = 5$

$dx = \frac{du}{5}$

$\int sin\ 5x\ dx + \int 6x\ \ dx$

$= \int sin\ u\ \frac{du}{5} + \frac{6x^2}{2} + c$

$= -\frac{1}{5} cos\ 5x + 3x^2 + c$

Example 3:
Integrate $cos\ 5x\ cos\ 5x\ -\ sin\ 5x\ sin\ 5x$.

$$\int\ cos\ 5x\ cos\ 5x\ -\ sin\ 5x\ sin\ 5x\ \ dx\ = \int cos(5x + 5x)dx$$

$$= \int cos\ 10x\ dx$$

Let $u = 10x,$

$\frac{du}{dx} = 10$

$dx = \frac{du}{10}$

$$\int \cos 10x \, dx$$
$$= \int \cos u \, dx$$
$$= \int \cos u \, \frac{du}{10}$$
$$= \frac{\sin u}{10} + c$$
$$= \frac{\sin 10x}{10} + c$$

Exponential functions

$$\int e^x \; dx = e^x + c$$

Example 1:
Integrate e^{2x}.

$$\int e^{2x} \; dx$$

Let $u = 2x$
$$\frac{du}{dx} = 2$$
$$dx = \frac{du}{2}$$

$$\int e^u \, \frac{du}{2}$$
$$= \frac{e^{2x}}{2} + c$$

Example 2:
Integrate $5e^{3x} + 3x$.

$$\int 5e^{3x} + 3x \; dx$$
Let $u = 3x$
$$\frac{du}{dx} = 3$$
$$dx = \frac{du}{3}$$

$$\int 5e^{3x} + 3x \; dx = \frac{5e^{3x}}{3} + \frac{3x^2}{2} + c$$

Example 3:
Integrate $6e^{3x+1}$.

$$\int 6e^{3x+1}\ dx$$

Let $u = 3x + 1$

$$\frac{du}{dx} = 3$$

$$dx = \frac{du}{3}$$

$$\int 6e^{3x^2+1}\ dx = \int 6e^{u}\ \frac{du}{3}$$
$$= 2e^{3x+1} + c$$

Logarithmic functions

$$\int \frac{1}{x}\ dx = \ln|x| + c$$
$$\int \frac{1}{ax+b}\ dx = ln(ax+b) + c$$

Example 1:
Integrate $\frac{7}{x}$.

$$\int \frac{7}{x}\ dx = 7\int \frac{1}{x}\ dx$$
$$= 7\ \ln x + c$$

Example 2:
Integrate $\frac{1}{6x}$.

$$\int \frac{1}{6x}\ dx = \frac{1}{6}\int \frac{1}{x}\ dx$$
$$= \frac{1}{6}\ln x + c$$

Example 3:
Integrate $\frac{1}{4x+5}$.

$$\int \frac{1}{4x+5}\ dx$$

Let $u = 4x + 5$

$$\frac{du}{dx} = 4$$

$$dx = \frac{du}{4}$$

$$\int \frac{1}{u}\ \frac{du}{4} = \frac{\ln(4x+5)}{4} + c$$

Example 4:
Integrate $\frac{4x}{4x^2+5}$.

$$\int \frac{4x}{4x^2+5}\,dx$$

Let $u = 4x^2 + 5$

$$\frac{du}{dx} = 8x$$

$$dx = \frac{du}{8x}$$

$$\int \frac{4x}{u}\frac{du}{8x} = \frac{\ln(4x^2+5)}{2} + c$$

Example 5:
Integrate $x + \frac{1}{x}$.

$$\int x + \frac{1}{x}\ dx = \frac{x^2}{2} + ln\, x + c$$

Example 6:
Integrate $tan\, 2\theta$.

$$\int tan\, 2\theta\ \ d\theta = \int \frac{sin\, 2\theta}{cos\, 2\theta} d\theta$$

Let $u = \ cos\, 2\theta$

$$\frac{du}{d\theta} = -2\, sin\, 2\theta$$

$$d\theta = \frac{du}{-2\, sin\, 2\theta}$$

$$\int \frac{sin\, 2\theta}{u}\frac{du}{-2\, sin\, 2\theta} = -\frac{1}{2}\int \frac{1}{u} du$$

$$= -\frac{1}{2}\ln cos\, 2\theta + c$$

Example 7:
Integrate $\frac{cos\, x}{sin\, x} + \frac{1}{x}$.

$$\int \frac{cos\, x}{sin\, x} + \frac{1}{x}\ dx = \ln sin\, x + \ln x + c$$

$$= \ln x\ sin\, x$$

2. Integration involving partial fraction

Cases for setting up a partial fraction

Case	Rational function	Partial fraction
Distinct linear factors	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
Distinct cubic linear factors	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}, a \neq b \neq c$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
Repeated linear factors	$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
	$\frac{px+q}{(x-a)^3}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$
Repeated linear and distinct linear factors	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$

Example 1:

Find the values of A, B and C given that $\frac{x^2+11}{(x+2)^2(x-3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-3}$. Hence, evaluate $\int_1^2 \frac{x^2+11}{(x+2)^2(x-3)}\, dx$.

$$\frac{x^2+11}{(x+2)^2(x-3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-3}$$

Multiply $(x+2)^2(x-3)$,

$x^2 + 11 = A(x+2)(x-3) + B(x-3) + C(x+2)^2$

Let $x - 3 = 0$ and $(x+2)^2 = 0$,

$\therefore x = 3$ and $x = -2$

When $x = 3$,

$3^2 + 11 = A(x+2)(3-3) + B(3-3) + C(3+2)^2$

$20 = 25C$

$C = \frac{4}{5}$

When $x = -2$,

$(-2)^2 + 11 = A(-2+2)(x-3) + B(-2-3) + C(-2+2)^2$

$15 = -5B$

$B = -3$

Equating coefficients of x^2,

$1 = A + C$

$1 = A + \frac{4}{5}$

$A = \frac{1}{5}$

$$\int_1^2 \frac{x^2 + 11}{(x+2)^2(x-3)}\, dx = \int_1^2 \frac{1}{5(x+2)} - \frac{3}{(x+2)^2} + \frac{4}{5(x-3)}\, dx$$

$$= [\frac{1}{5}\ln(x+2) + \frac{3}{x+2} + \frac{4}{5}\ln(x-3)]_1^2$$

$$= \frac{4\ln(12) + 5}{20}$$

Example 2:

Find the values of A and B given that $\frac{3-x}{5+3x-2x^2} = \frac{A}{5-2x} + \frac{B}{1+x}$. Hence evaluate $\int_0^2 \frac{3-x}{5+3x-2x^2}\, dx$.

$$\frac{3-x}{5+3x-2x^2} = \frac{A}{5-2x} + \frac{B}{1+x}$$

Multiply $(5-2x)(1+x)$,

$3 - x = A(1+x) + B(5-2x)$

Let $(5-2x)(1+x) = 0$

$\therefore x = \frac{5}{2}$ and $x = -1$

When $x = \frac{5}{2}$,

$$3 - \frac{5}{2} = A\left(1 + \frac{5}{2}\right) + B\left(5 - 2 \times \frac{5}{2}\right)$$

$$\frac{1}{2} = 3\frac{1}{2}A$$

$$A = \frac{1}{7}$$

When $x = -1$,

$$3 - (-1) = A(1-1) + B[5 - 2(-1)]$$

$$4 = 7B$$

$$B = \frac{4}{7}$$

$$\int_0^2 \frac{3-x}{5+3x-2x^2}\, dx = \int_0^2 \frac{1}{7(5-2x)} + \frac{4}{7(1+x)}\, dx$$

$$= [\frac{1}{7}\frac{\ln(5-2x)}{-2} + \frac{4}{7}\ln(1+x)]_0^2$$

$$= [-\frac{\ln(5-2x)}{14} + \frac{4}{7}\ln(1+x)]_0^2$$

$$= \frac{4}{7}\ln 3 + \frac{1}{14}\ln 5$$

3.	**The arbitrary constant, "c" in indefinite integration**
	(a) Origin

Origin of arbitrary constant (by example):
By differentiating $y = mx + c$, we can get $\frac{dy}{dx} = m$. The value of c disappears as it does not have an unknown, x.

$$y = 3x + 1$$
$$y = 3x + 2$$
$$y = 3x + 3$$
$$.....$$

For all the equations above,

$$\frac{dy}{dx} = 3$$

If we integrate $\frac{dy}{dx} = 3$,

$$\int 3\,dx = 3x$$

From here, we can see that the equation is $y = 3x$. However, there should be a constant as

$$y = 3x + 1 \quad \neq \quad y = 3x + 2 \quad \neq \quad y = 3x + 3$$

Therefore, the integration of these equations should give $y = 3x + c$ where c is a constant, $c = 1{,}2{,}3$ for this case.

(b) How different ways of integration affects arbitrary constant

Example 1:

Method 1

$$\int cos^3x\, sin\, x\, dx = \int cos\, x\, cos^2x\, sin\, x \;\; dx$$
$$= \int (1 - sin^2x)\, cos\, x\, sin\, x\, dx$$

Let $u = sin\, x$,

$$\frac{du}{dx} = cos\, x$$

$$\int (1 - u^2)\, u\, cos\, x\, \frac{du}{cos\, x}$$
$$= \int u - u^3 \;\; du$$
$$= \frac{u^2}{2} - \frac{u^4}{4} + c$$
$$= \frac{sin^2x}{2} - \frac{sin^4x}{4} + c$$

$= \frac{1-cos^2x}{2} - \frac{(1-cos^2x)^2}{4} + c$

$= \frac{1-cos^2x}{2} - \frac{1-2cos^2x+cos^4x}{4} + c$

$= \frac{1}{2} - \frac{cos^2x}{2} - \frac{1}{4} + \frac{cos^2x}{4} - \frac{cos^4x}{4} + c$

$= -\frac{cos^4x}{4} + \frac{1}{4} + c$

<u>Method 2</u>

$\int cos^3x \, sin \, x \, dx$

Let $u = cos \, x$,

$\frac{du}{dx} = -sin \, x$

$\int u^3 \, sin \, x \; -\frac{du}{sin \, x}$

$= \int -\frac{u^4}{4} \, du$

$= -\frac{cos^4x}{4} + C$

Both answers are correct. By comparing both answers,

$$C = \frac{1}{4} + c$$

Or also

$$c = C - \frac{1}{4}$$

Example 2:

<u>Method 1</u>

$\int x + 1 \, dx = \frac{x^2}{2} + x + c$

<u>Method 2</u>

$\int x + 1 \, dx$

Let $u = x + 1$

$\int u \, du$

$= \frac{u^2}{2} + C$

$= \frac{(x+1)^2}{2} + C$

$= \frac{x^2}{2} + x + \frac{1}{2} + C$

	Both answers are correct. By comparing both answers, $$c = \frac{1}{2} + C$$ Or also $$C = c - \frac{1}{2}$$ Example 3: <u>Method 1</u> $$\int \frac{7}{5x}\,dx = \frac{7}{5}\int \frac{1}{x}\,dx$$ $$= \frac{7}{5}\ln x + c$$ <u>Method 2</u> $$\int \frac{7}{5x}\,dx = \frac{7}{5}\int \frac{5}{5x}\,dx$$ $$= \frac{7}{5}\ln 5x + C$$ $$= \frac{7}{5}\ln 5 + \frac{7}{5}\ln x + C$$ $$= \frac{7}{5}\ln x + \frac{7}{5}\ln 5 + C$$ Both answers are correct. By comparing both answers, $$c = \frac{7}{5}\ln 5 + C$$ Or also $$C = c - \frac{7}{5}\ln 5$$
4.	**Finding equation of a curve**
	Example 1: Find the equation of curve passing with gradient function $f'(x) = 5x^2 + 2x$ at $(3,5)$. $$\int 5x^2 + 2x\ \,dx = \frac{5x^3}{3} + x^2 + c$$ $$y = \frac{5x^3}{3} + x^2 + c$$ At $(3,5)$, $$5 = \frac{5(3^3)}{3} + 3^2 + c$$ $$c = 5 - 54$$ $$= -49$$ Equation is $y = \frac{5x^3}{3} + x^2 - 49$

Example 2:

Find v given that $\frac{dv}{dt} = \frac{50t}{(t^2-1)^2}$ at $(2,3)$.

$$\int \frac{50t}{(t^2-1)^2}\ dt$$
$$= \int 50\,t\,(t^2-1)^{-2}\,dt$$
Let $u = t^2 - 1$

$$\frac{du}{dt} = 2t$$
$$dt = \frac{du}{2t}$$
$$\int 50\,t\,(u)^{-2}\,dt$$
$$= \int 50\,t\,(u)^{-2}\,\frac{du}{2t}$$
$$= \frac{25\,u^{-2+1}}{-2+1} + c$$
$$= -\frac{25\,(t^2-1)^{-1}}{1} + c$$
$$= -\frac{25}{t^2-1} + c$$

At $(2,3)$,
$$3 = -\frac{25}{(2)^2-1} + c$$
$$c = \frac{34}{3}$$

$$v = -\frac{25}{t^2-1} + \frac{34}{3}$$

Example 3:

The tangent to the curve $y = f(x)$ at point $(2,0)$ is equated by $y = 2x + 3$. The gradient function is $f'(x) = zx + h$. What is the equation of curve that it passes through $(4,7)$?

$$f'(x) = zx + h$$
$$\int zx + h\,dx = \frac{zx^2}{2} + hx + c$$

Equation of curve is $f(x) = \frac{zx^2}{2} + hx + c$

At $(4,7)$,

$7 = \frac{16z}{2} + 4h + c$

$7 = 8z + 4h + c$ ......*(1)*

At $(2,0)$,

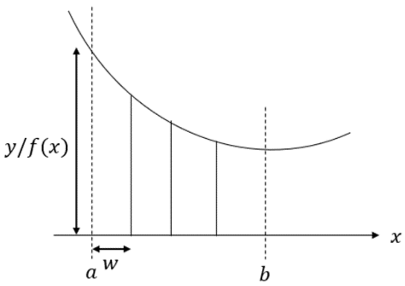
$0 = \frac{4z}{2} + 2h + c$

$0 = 2z + 2h + c$ ......*(2)*

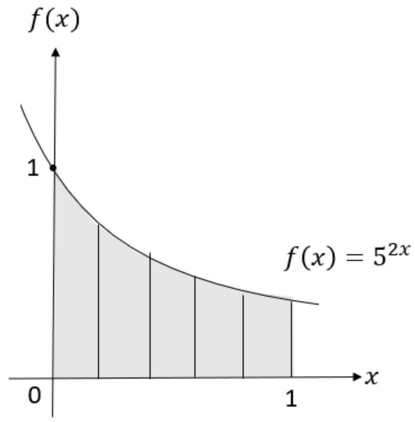
(1)−(2),

$7 = 6z + 2h$

$6z + 2h = 7$ ....*(3)*

	$f'(x) = zx + h$ $2 = zx + h$ *At* $(2,0)$, $2z + h = 2$ ....*(4)* *(4)*$\times$ 2, $4z + 2h = 4$ .....*(5)* *(3)–(5)*, $2z = 3$ $z = \frac{3}{2}$ *When* $z = \frac{3}{2}$, $2(\frac{3}{2}) + h = 2$ $3 + h = 2$ $h = -1$ $f(x) = \frac{zx^2}{2} + hx + c$ $= \frac{\frac{3}{2}x^2}{2} - x + c$ $= \frac{3x^2}{4} - x + c$ *At* $(2,0)$, $0 = \frac{3(2)^2}{4} - 2 + c$ $c = -1$ *Therefore,* $f(x) = \frac{3x^2}{4} - x - 1$
5.	**Area under the curve**
	(a) Trapezium rule
	Additional info. Given a curve with function $f(x)$  To find each area of strips (trapezium): $$Area = \frac{1}{2}(y_0 + y_1)\, w$$ Total area under the curve by calculating the total area of rectangular strips, $$Area\ = w\,[\frac{1}{2}(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots y_{n-1})]$$

Example:
Diagram below shows a function $f(x) = 5^{2x}$.



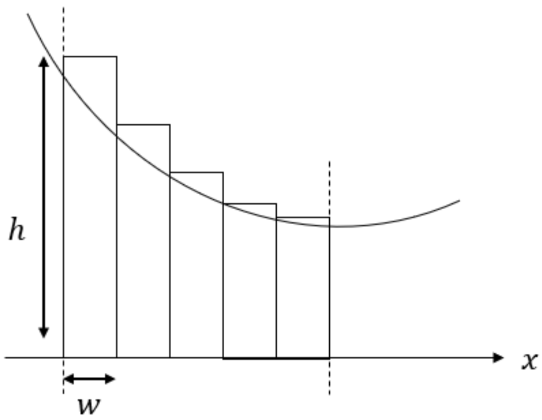
Estimate the shaded area using trapezium rule.

$$Width\ of\ trapezium\ strips = \frac{1}{5} = 0.2$$

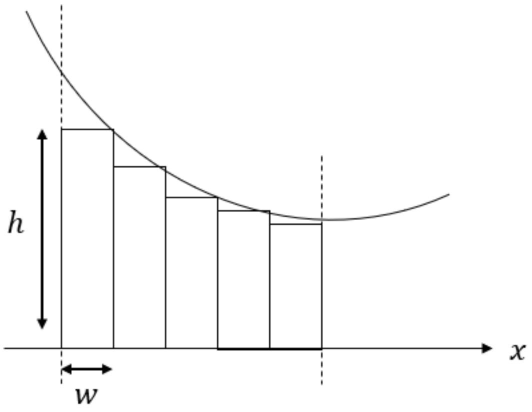
$$A \approx \frac{1}{2}(0.2)\{[f(0) + f(0.2)] + [f(0.2) + f(0.4)] + [f(0,4) + f(0.6)] + [f(0.6) + f(0.8)] + [f(0.8) + f(1)]\}$$
$$\approx \frac{1}{2}(0.2)[\,f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)]$$
$$\approx 7.712\ units^2$$

(b) Rectangle method/ midpoint rule

Given a curve with function $f(x)$



Area circumscribed



Area inscribed

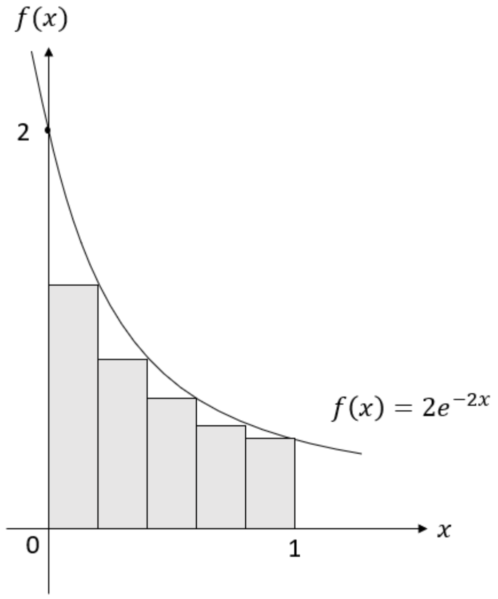
To find each area of strips (rectangles):

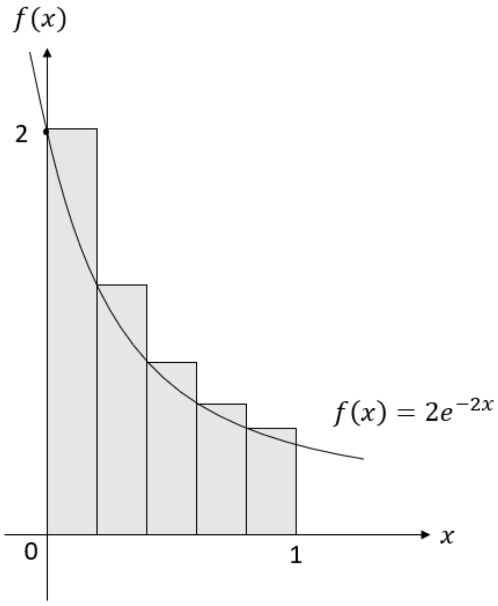
$$A = f(x)/\ h \times w$$

Estimating the area of curve,

$$Area \approx \frac{A_{circumscribed} + A_{inscribed}}{2}$$

Example:
Diagrams below shows graphs of $f(x) = 2e^{-2x}$ inscribed and circumscribed.





Estimate the area of the region trapped between the curve and x −axis from $x = 0$ to $x = 1$.

$$Width\ of\ strips = \frac{1}{5}$$
$$= 0.2$$

$$A = f(x)/\ h \times w$$

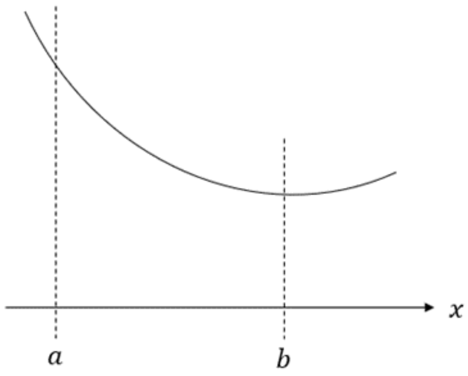
$$Total\ area\ inscribed = w[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)]$$
$$= 0.2[3.516]$$
$$= 0.7032$$

$$Total\ area\ circumscribed = w[f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)]$$
$$= 0.2[5.245]$$
$$= 1.049$$

$$Area \approx \frac{A_{circumscribed} + A_{inscribed}}{2}$$
$$\approx \frac{0.7032 + 1.049}{2}$$
$$\approx 0.8761\ units^2$$

(c) Integration (definite integral)

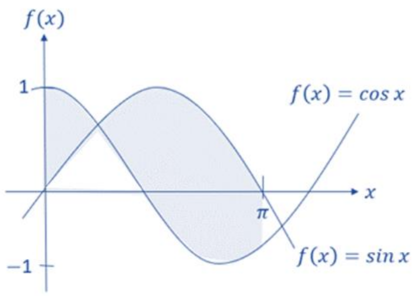
Given a curve with function $f(x)$



$$Area = \int_a^b f(x)\,dx$$

Tips for finding area bounded by two functions:
Use the function above minus the function below.

Example:
Find the area trapped between $f(x) = sin\ x$ and $f(x) = cos\ x$ for the range $0 \le x \le \pi$.



Graph intersects at $x = \frac{\pi}{4}$ *in the range* $0 \le x \le \pi$

$$A = \int_0^{\frac{\pi}{4}} cos\ x - sinx\ dx + \int_{\frac{\pi}{4}}^{\pi} sin\ x - cosx\ dx$$
$$= 2\sqrt{2}\ units^2$$

6.	**Fundamental theorem of calculus**
	(a) Evaluation theorem: Part 2
	$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$
	Example: Use the fundamental theorem of calculus to evaluate $\int_0^1 x^2 + e^x \ dx$. Give your answer in terms of e. $\int_0^1 x^2 + e^x \ dx = [\frac{x^3}{3} + e^x]_0^1$ $= [\frac{1}{3} + e] - [\frac{0}{3} + e^0]$ $= -\frac{2}{3} + e$
	(b) Relationship between differentiation and integration: Part 1
	Finding derivative using fundamental theorem of calculus $$\frac{d}{dx}[\int_a^x f(t)]\, dt = f(x)$$
	Example 1: Determine $\frac{d}{dx}[\int_1^x t^2 + 2]\, dt$. $$\frac{d}{dx}[\int_1^x t^2 + 2]\, dt = x^2 + 2$$ Example 2: Determine $\frac{d}{dy}[\int_1^y 3t^5 + 2t]\, dt$. $$\frac{d}{dy}[\int_1^y 3t^5 + 2t]\, dt = 3y^5 + 2y$$
	Using fundamental theorem & chain rule to calculate derivatives $$\frac{d}{dx}[\int_a^{g(x)} f(t)]\, dt = f[g(x)] \times g'(x)$$
	Example 1: Find $\frac{d}{dx}[\int_1^{x+1} t]\, dt$. $\frac{d}{dx}[\int_1^{x+1} t]\, dt = (x+1) \times \frac{d}{dx}(x+1)$ $= x + 1$

Example 2:

Find $\frac{d}{dx}\ [\int_{\pi}^{e^x} t^2 + t]\ dt.$

$$\frac{d}{dx}\ [\int_{\pi}^{e^x} t^2 + t]\ dt = (e^{2x} + e^x) \times \frac{d}{dx} e^x$$
$$= (e^{2x} + e^x)e^x$$

Using fundamental theorem of calculus with two variable limits of integration

Steps:

1. Break the integrals in accordance to $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$
2. Apply $\frac{d}{dx}\ [\int_a^{g(x)} f(t)]\ dt = f[g(x)] \times g'(x)$ and/ or $\frac{d}{dx}\ [\int_a^x f(t)]\ dt = f(x)$ whenever necessary.

Example 1:

Find $f'(x)$ of $f(x) = \int_t^{3t} x^3\ dx.$

$$\frac{d}{dt}\int_t^{3t} x^3\ dx = \frac{d}{dt}\int_0^{3t} x^3\ dx + \frac{d}{dt}\int_t^{0} x^3\ dx$$
$$= (3t)^3 \times \frac{d}{dt}(3t) - \frac{d}{dt}\int_0^{t} x^3\ dx$$
$$= 3(27t^3) - t^3$$
$$= 80t^3$$

Example 2:

Find $\frac{d}{dx}[\int_{x+2}^{\ln 2x} y^2\ dy].$

$$\frac{d}{dx}[\int_{x+2}^{\ln 2x} y^2 dy\,] = \frac{d}{dx}[\int_0^{\ln 2x} y^2\ dy] + \frac{d}{dx}[\int_{x+2}^{0} y^2\ dy]$$
$$= (\ln 2x)^2 \times \frac{d}{dx}(\ln 2x) - \frac{d}{dx}[\int_0^{x+2} y^2\ dy]$$
$$= \frac{(\ln 2x)^2}{x} - (x+2)^2 \times \frac{d}{dx}(x+2)$$
$$= \frac{(\ln 2x)^2}{x} - (x+2)^2$$

Theorem (iii)

$$\int_b^a \frac{d}{dt}[f(t)]\ dt = f(a) - f(b)$$

Example 1:

Find $\int_2^x \frac{d}{dt}(t^3 + 1)dt$.

$$\int_2^x \frac{d}{dt}(t^3 + 1)dt = [x^3 + 1] - [2^3 + 1]$$
$$= x^3 - 8$$

Example 2:

Find $\int_{\pi}^{x^2} \frac{d}{dt}(2t^2+t)dt$.

$$\int_{\pi}^{x^2} \frac{d}{dt}(2t^2+t)dt = [2(x^2)^2+x^2]-[2\pi^2+\pi]$$
$$= 2x^4+x^2-2\pi^2-\pi$$

7. Additivity and linearity of definite integrals

Summary:

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

$$\int_a^b k \times f(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$

$$\int_a^a f(x)dx = 0$$

Example:

Given that $\int_1^7 f(x)\, dx = 3$, evaluate $\int_7^7 2f(x)\, dx$.

$$\int_7^7 2f(x)\, dx = 0$$

Using substitution method

Example:

Given that $f(x)$ is continuous everywhere and that $\int_7^{15} f(x)\, dx = 7$, evaluate $\int_2^{10} f(x+5)\, dx$.

$let\ u = x+5,$

$\frac{du}{dx} = 1$

$du = dx$

$When\ x = 10,$

$u = 15$

$When\ x = 2,$

$u = 7$

$$\int_2^{10} f(x+5)\, dx = \int_7^{15} f(u)\, du$$
$$= 7$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Example 1:

Given that $\int_1^{100} f(x)dx = e^{12}$, evaluate $\int_{100}^{1} f(x)dx$

$$\int_{100}^{1} f(x)dx = -\int_1^{100} f(x)dx$$
$$= -e^{12}$$

Example 2:

Given that $\int_{-10}^{-2} f(x)dx = 5$, evaluate $\int_{10}^{2} f(-x)dx$.

$let\ u = -x,$

$$\frac{du}{dx} = -1$$

$-du = dx$

$When\ x = 10,$
$u = -10$
$When\ x = 2,$
$u = -2$

$$\int_{10}^{2} f(-x)dx = \int_{-10}^{-2} f(u) - du$$
$$= -5$$

$$\int_a^b k \times f(x)dx = k\int_a^b f(x)dx$$

Example 1:

Given that $\int_1^7 f(x)\,dx = 3$, evaluate $\int_1^7 7f(x)\,dx$.

$$\int_1^7 7f(x)dx = 4(3)$$
$$= 28$$

Example 2:

Given that $\int_3^7 f(x)\,dx = 12$, evaluate $\int_3^7 \frac{f(x)}{4}\,dx$.

$$\int_3^7 \frac{f(x)}{4}\,dx = \frac{12}{4}$$
$$= 3$$

Example 3:

Given that $\int_5^{12} f(x)\,dx = 7$, evaluate $\int_1^7 2f(x+3)\,dx$.

$let\ u = x + 3,$

$\frac{du}{dx} = 1$

$du = dx$

$When\ x = 9,$
$u = 12$

$When\ x = 2,$
$u = 5$

$$\int_1^7 2f(x+3)dx = \int_5^{12} 2f(u)\,du$$
$$= 2(7)$$
$$= 14$$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

Example 1:

Given that $\int_{12}^{6} f(x)dx = 100$, evaluate $\int_{12}^{18} f(x)\,dx - \int_6^{18}[f(x)+10]\,dx$.

$$\int_{12}^{18} f(x)\,dx - \int_6^{18}[f(x)+10]\,dx = \int_{12}^{18} f(x)\,dx - \int_6^{18} f(x)\,dx - \int_6^{18} 10\,dx$$
$$= \int_{12}^{18} f(x)\,dx + \int_{18}^{6} f(x)\,dx - [10x]_6^{18}$$
$$= \int_{12}^{6} f(x)\,dx - 120$$
$$= 100 - 120$$
$$= -20$$

Example 2:

Given that $\int_1^7 f(x)dx = 13$ and $\int_7^6 f(x)dx = 24$, evaluate $\int_1^7 f(x)dx + \int_6^7 f(x)dx$.

$$\int_1^7 f(x)dx + \int_6^7 f(x)dx = 13 + (-\int_7^6 f(x)dx)$$
$$= 13 - 24$$
$$= -11$$

	$$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$ $$\int_a^b [f(x) \pm c\,]\, dx = \int_a^b f(x)\, dx \pm \int_a^b c\, dx$$
	Example 1: Given that $\int_9^{87} f(x)\, dx = -43$, evaluate $\int_9^{87} [f(x) + 10]\, dx$. $\int_9^{87} [f(x) + 10]\, dx = \int_9^{87} 10\, dx + (-43)$ $= [10x]_9^{87} - 43$ $= 780 - 43$ $= 737$ Example 2: Given that $\int_1^9 f(x)\, dx = 3.5$, evaluate $\int_1^9 [f(x) - 10x]\, dx$. $\int_1^9 [f(x) - 10x]\, dx. = 3.5 - [\frac{10x^2}{2}]_1^9$ $= 3.5 - 400$ $= -396.5$

END